Complexes, Convex Geometries, and Orders

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The Representation Theorem

Let $(P, \leq)$ be a poset of dimension d. For $A \subseteq P$ let

$$\mathrm{Inf}(A) = \{v \in P \mid (\forall a \in A)\ v \leq a\}$$

Theorem

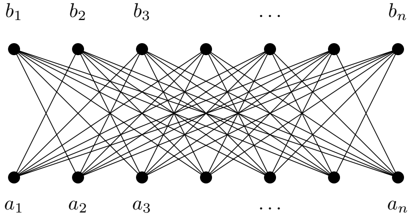
There exists $f : P \to \mathbb{R}^{d-1}$ such that for every $A, B \subseteq P$ we have

$$\mathrm{Conv}(f(\mathrm{Inf}(A) \setminus \mathrm{Inf}(B)) \cap \mathrm{Conv}(f(\mathrm{Inf}(B) \setminus \mathrm{Inf}(A)) = \emptyset.$$

Remark

Tight.

The Standard Example



- Assume $f : S_n \to \mathbb{R}^{n-2}$.

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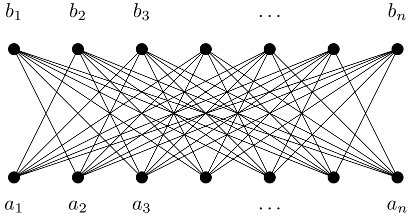

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- Radon partition (U, V) of $\{f(a_1), \ldots, f(a_n)\}$: $\mathrm{Conv}(U) \cap \mathrm{Conv}(V) \neq \emptyset$.

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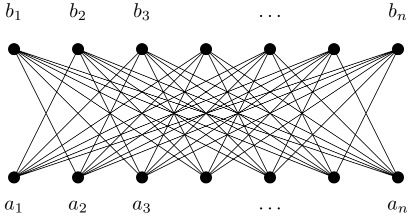

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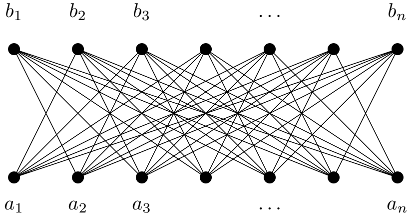

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- Radon partition (U, V) of $\{f(a_1), \ldots, f(a_n)\}$: $\mathrm{Conv}(U) \cap \mathrm{Conv}(V) \neq \emptyset$.
- Let $A = \{b_i : f(a_i) \in U\}, B = \{b_i : f(a_i) \in V\}$.
- Then $\mathrm{Inf}(A) = V$ and $\mathrm{Inf}(A) = W$ $\lightning$

Proof of the Theorem ($\mathbb{R}^d$)

$<_1, \ldots, <_d$ realizer;
$F_i : P \to [1, +\infty)$ s.t. $x >_i y \Rightarrow F_i(x) > d\, F_i(y)$.

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$$L_{A,B}(\mathbf{x}) = \sum_{i:\ \min A <_i \min B} \frac{x_i}{\min_{a \in A} F_i(a)}.$$

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  $\Rightarrow\ L_{A,B}(F(z)) \leq d$.

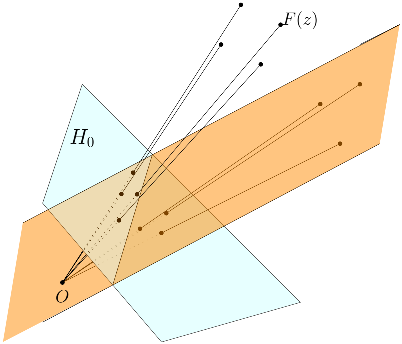
$\Rightarrow$ separation with hyperplane $L_{A,B}(\mathbf{x}) - L_{B,A}(\mathbf{x}) = 0$.

Proof (cont'd)

Observations:

- The origin O belongs to every hyperplane $L_{A,B}(\mathbf{x}) - L_{B,A}(\mathbf{x}) = 0$;
- The hyperplane $H_0 : \sum x_i = 1$ separates O from $F(P)$.

$x \mapsto H_0 \cap (O, F(z))$!



Simplicial Complex

Definition

Δ is an *abstract simplicial complex* if $X \in \Delta$ and $Y \subseteq X$ imply $Y \in \Delta$.

Theorem (POM '99)

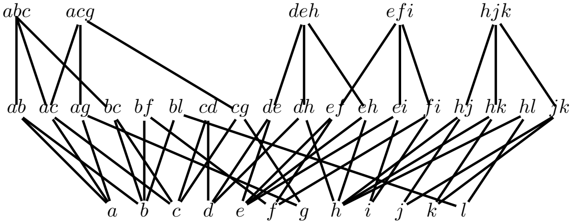
Every abstract simplicial complex Δ has a geometric realization in $\mathbb{R}^{d-1}$, where $d = \dim P(\Delta)$.

Proof.

Sufficient to prove separation of disjoint faces. □

Abstract Simplicial Complex

Consider $\mathcal{F} = \{abc, acg, deh, efi, hjk, cd, bf, bl, hl\}$.



$<_1$: $a\ b\ ab\ c\ ac\ bc\ abc\ d\ cd\ e\ de\ f\ bf\ ef\ i\ ei\ fi\ efi\ h\ dh\ eh\ deh\ g\ ag\ cg\ acg\ j\ hj\ k\ hk\ jk\ hjk\ l\ bl\ hl$
$<_2$: $b\ l\ bl\ i\ h\ hl\ k\ hk\ j\ hj\ jk\ hjk\ g\ f\ bf\ fi\ e\ ei\ eh\ ef\ efi\ d\ dh\ de\ deh\ c\ bc\ cg\ cd\ a\ ab\ ag\ ac\ abc\ acg$
$<_3$: $l\ a\ g\ ag\ c\ ac\ cg\ acg\ d\ cd\ k\ j\ jk\ h\ hl\ dh\ hk\ hj\ hjk\ e\ de\ eh\ deh\ i\ ei\ f\ ef\ fi\ efi\ b\ bl\ ab\ bc\ abc\ bf$

$<_1$: $a\ b\ c\ d\ e\ f\ i\ h\ g\ j\ k\ l$
$<_2$: $b\ l\ i\ h\ k\ j\ g\ f\ e\ d\ c\ a$
$<_3$: $l\ a\ g\ c\ d\ k\ j\ h\ e\ i\ f\ b$

Retrieving Faces

- deh is a face so $\forall x \notin \{d, e, h\}\ \exists i\ x >_i deh >_i d, e, h$;
- de is a face so $\exists i\ h >_i de >_i d, e$.

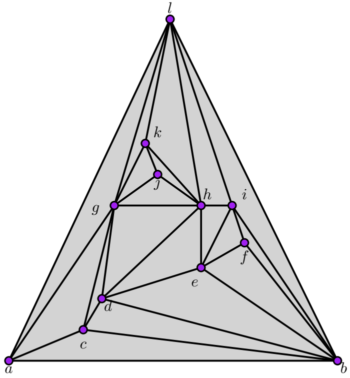
Generally, F is a face $\rightarrow \forall x\ \exists i\ x \geq_i \max_i F$.

$$
\begin{array}{l}
<_1\colon a\ b\ c\ \boxed{d}\ \boxed{e}\ f\ i\ \boxed{h}\ g\ j\ k\ l \\
<_2\colon b\ l\ i\ \boxed{h}\ k\ j\ g\ f\ \boxed{e}\ \boxed{d}\ c\ a \\
<_3\colon l\ a\ g\ c\ \boxed{d}\ k\ j\ \boxed{h}\ \boxed{e}\ i\ f\ b
\end{array}
$$

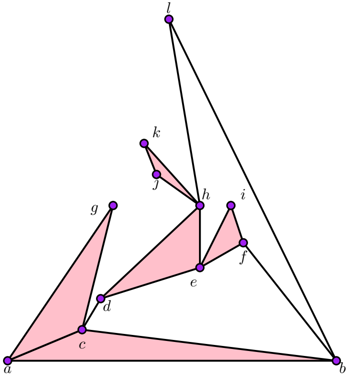
Triangulating

Defining faces by $\forall x\ \exists i\ x \geq_i \max_i F$ we get a triangulation $\widehat{\Delta} \supseteq \Delta$.

$\widehat{\Delta}$



Δ



Convex Geometry

Definition

Closure operator $\phi : \mathcal{P}(U) \to \mathcal{P}(U)$ s.t. $\phi(\emptyset) = \emptyset$ and

$$A \subseteq \phi(A), \qquad A \subseteq B \;\Rightarrow\; \phi(A) \subseteq \phi(B), \qquad \phi(\phi(A)) = \phi(A)$$

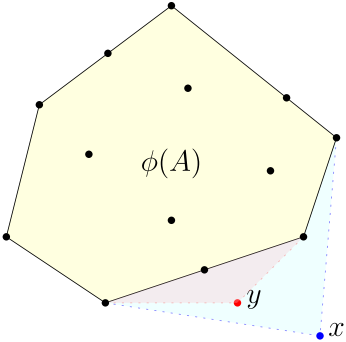
Anti-exchange axiom:

$$x \neq y,\; x, y \notin \phi(A),\; y \in \phi(A + x) \quad \Longrightarrow \quad x \notin \phi(A + y)$$

Convex geometry:
closure operator ϕ with anti-exchange property.

The anti-exchange property

$$x \neq y,\ x, y \notin \phi(A),\ y \in \phi(A + x) \quad \Longrightarrow \quad x \notin \phi(A + y)$$



Characterizations

Theorem (Björner '85; Edelman and Jamison '85; Korte and Lovász '84)

Let $\phi : \mathcal{P}(U) \to \mathcal{P}(U)$ be a closure operator. Then

$$\begin{aligned}
& \phi \text{ has the anti-exchange property} \\
\iff & \forall \text{ convex } K \neq U, \exists p \notin X \text{ s.t. } K + p \text{ is convex} \\
\iff & \text{every } A \subseteq U \text{ has a unique basis} \\
\iff & \forall \text{ convex } K,\ K = \phi(\text{ex}(K)) \\
\iff & \forall \text{ convex } K \text{ and } p \notin K,\ p \in \text{ex}(\phi(K + p)) \\
\iff & \forall p \text{ and copoint } C \text{ attached to } p,\ C + p \text{ is convex.}
\end{aligned}$$

convex : $K = \phi(A)$ for some A

basis : minimal $B \subseteq A$ s.t. $A = \phi(B)$

ex : $\text{ex}(A) = \{p \in A \mid p \notin \phi(A - p)\}$

copoint : maximal convex $C \not\ni p$

Other Axiomatization

A π-system $\mathcal{C} \supseteq \{\emptyset, U\}$ is the family of convex sets of a convex geometry if

$$\forall K \in \mathcal{C} - U \quad \exists p \in U \setminus K \quad K + p \in \mathcal{C}.$$

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Assume $\forall p \in U, \{p\} \in \mathcal{C}$ and realizer $<_1, \ldots, <_d$ of $(\mathcal{C}, \subseteq)$.
Then $K \in \mathcal{C}$ implies

$$\forall x \notin K \quad \exists i \quad \{x\} >_i K$$

Completion

Define $\phi : \mathcal{P}(U) \to \mathcal{P}(U)$ by

$$\phi(A) = \{v \mid \forall i \ v <_i \max A\}.$$

$\to$ convex geometry with family of convex sets $\widehat{\mathcal{C}} \supseteq \mathcal{C}$.

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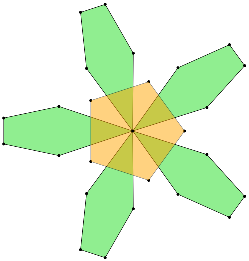
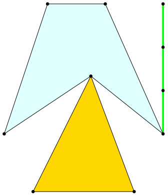
$\to$ $\widehat{\mathcal{C}}$ includes a triangulated simplicial complex $\widehat{\Delta}$
$\rightsquigarrow$ each $K \in \widehat{\mathcal{C}}$ is a subcomplex of $\widehat{\Delta}$.

NO



BUT



Lattice of Convex Sets

Lattice of Convex Sets (Edelman '80)

$$K_1 \wedge K_2 = K_1 \cap K_2$$
$$K_1 \vee K_2 = \phi(K_1 \cup K_2)$$

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Possible to extend to $\mathcal{P}(U)$?

$$X \wedge_\phi Y = \begin{cases} \phi(X) \cap \phi(Y) & \text{if } \phi(X) \neq \phi(Y) \\ \mathrm{ex}(X) \cup ((X - \mathrm{ex}(X)) \wedge_\phi (Y - \mathrm{ex}(Y)) & \text{otherwise} \end{cases}$$

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$\bowtie$-free height 2 Posets

Assume $(P, \leq)$ is $\bowtie$-free, has height 2, and every maximal element covers at least 2 elements.
Let $<_1, \ldots, <_d$ be a realizer of P.
We can interpret minimal elements as U and maximal elements as a family $\mathcal{M}$ of incomparable subsets of U.

$\rightarrow$ $\forall$ maximal S and $\forall x \notin S$ $\exists i$ such that $x >_i S >_i \max S$.

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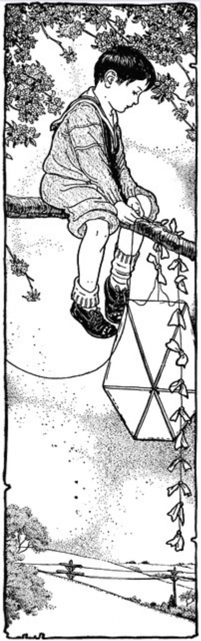
- → $\forall$ maximal S and $\forall x \notin S$ $\exists i$ such that $x >_i S >_i \max S$.
- → $\mathcal{M} \subset \widehat{\mathcal{C}}$
- → representation as a contact system of complexes.



Thank you for your attention.