Dimension of contact systems of d-dimensional boxes

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Order & Geometry
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Definitions

A *box* B in $\mathbb{R}^d$: $[a_1, b_1] \times \ldots \times [a_d, b_d]$.
$dim(B)$: number of non-trivial intervals $[a_i, b_i]$.

Contact system $\mathscr{B}$ of boxes in $\mathbb{R}^d$

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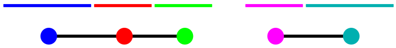
Contact complex $K(\mathscr{B})$

A simplicial complex with vertex set $\mathscr{B}$ and where $F \subseteq \mathscr{B}$ is a face if and only if the boxes of F intersect.

Dimension 1 and 2

In $\mathbb{R}^1$:
$K(\mathscr{B}) = G(\mathscr{B})$ is a path forest



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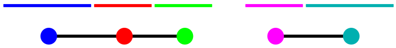

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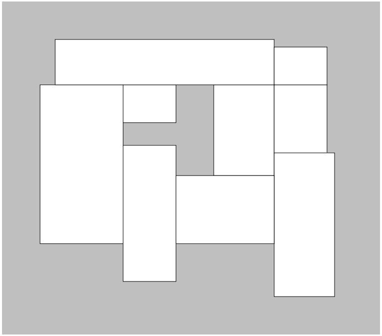

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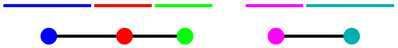

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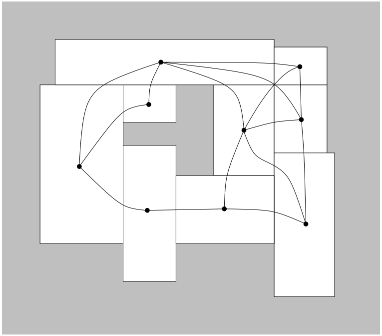

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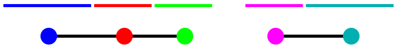
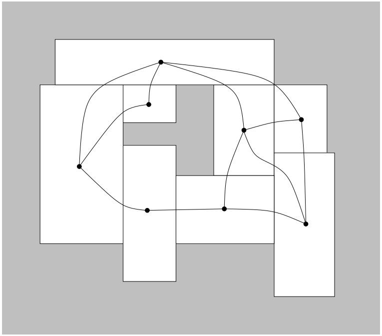


In $\mathbb{R}^2$:

Theorem (Thomassen '84)

$G = G(\mathscr{B})$ for some $\mathscr{B}$ with no 4 boxes intersection $\Longleftrightarrow$ G is a proper subgraph of a 4-conn. planar graph.



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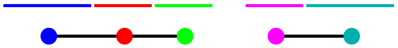

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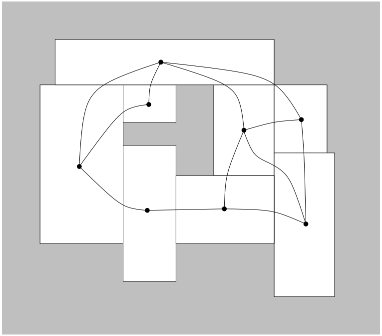
Theorem (Thomassen '84)

$K = K(\mathscr{B})$ for some $\mathscr{B}$ with no 4 boxes intersection $\iff$ K is a *clique complex* and *embeds* in the plane.



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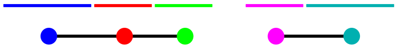

$dim_{DM}(G(\mathscr{B})) = 2$

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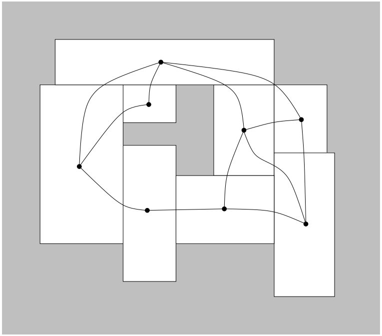

$dim_{DM}(G(\mathscr{B})) = dim_{DM}(K(\mathscr{B})) \leq 3$

Dushnik-Miller dimension

Dushnik-Miller dimension of a simplicial complex

$dim_{DM}(K) =$ Min k s.t. $\exists \leq_1 \ldots \leq_k$ total orders on $V(K)$ s.t.
$\forall F \in K, \forall x \in V(K), \exists i$ s.t. $F \leq_i x$ $(i.e. \forall y \in F\, y \leq_i x)$

Remark

$dim_{DM}(K) = dim(\mathscr{I}(K))$, where $\mathscr{I}(K)$ is the inclusion poset of K.

Example : the path $abcd$:

$\leq_1$: $a \leq_1 b \leq_1 c \leq_1 d$

$\leq_2$: $d \leq_2 c \leq_2 b \leq_2 a$

Example : Empty triangle & empty rectangle complexes

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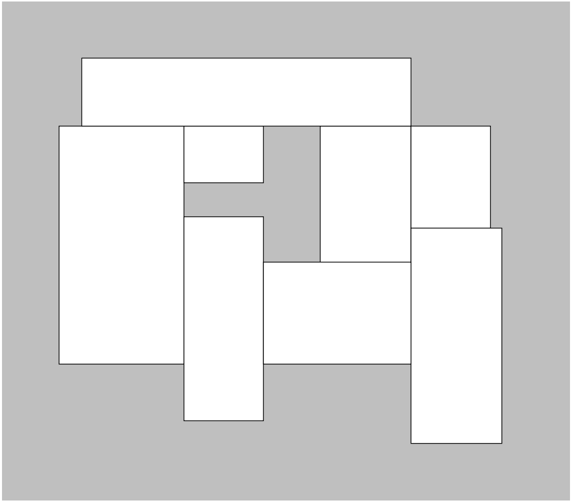
Theorem (Scarf '73, Ossona de Mendez '99)

Simplicial complexes of DM-dimension d linearly embed in $\mathbb{R}^{d-1}$.

Construction in $\mathbb{R}^2$ (from Schnyder Woods by Zhang '10)



Intersection btw 2 boxes
$\Rightarrow$ 1-dimensional

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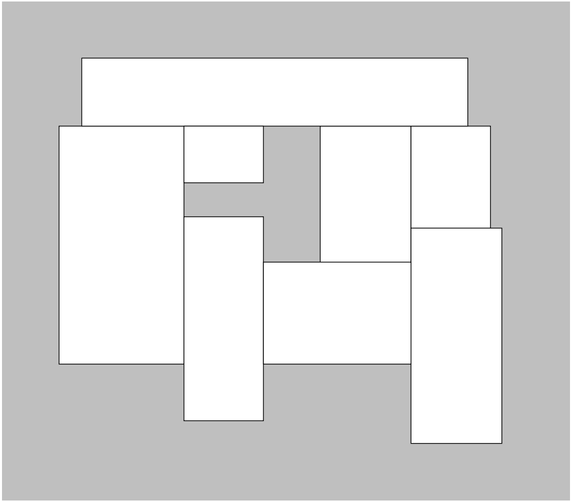

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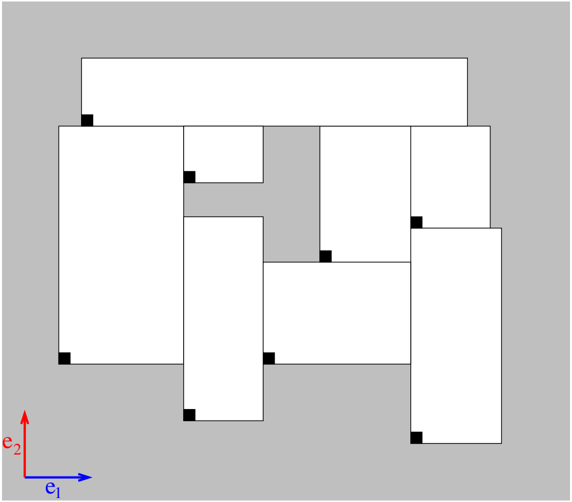

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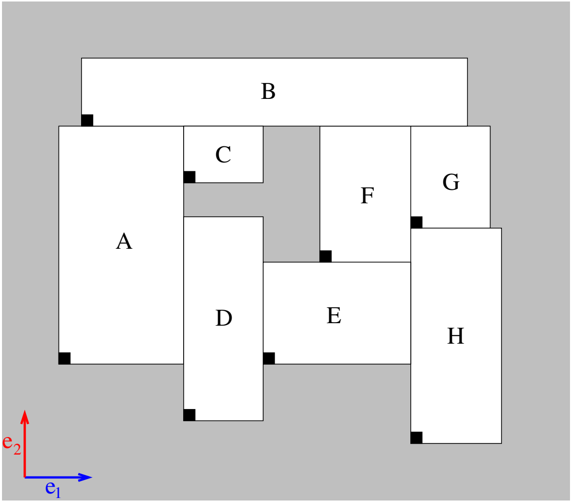

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follow e_1, and $-e_2$ if $=$
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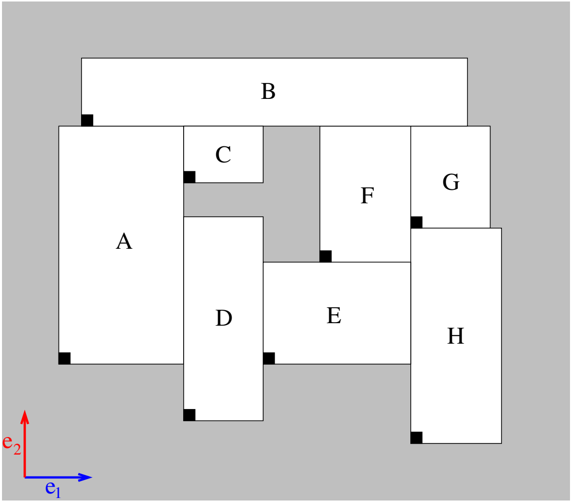

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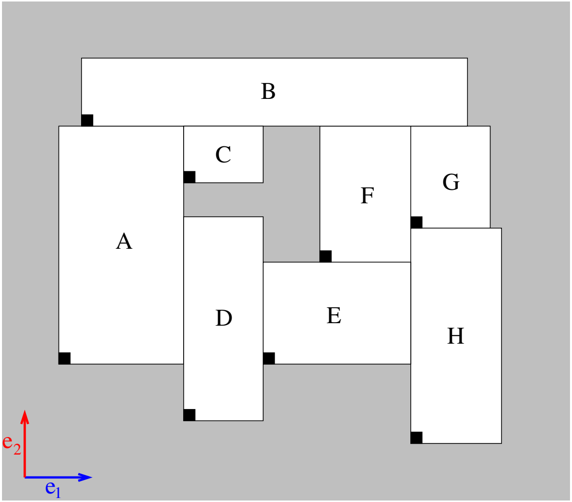

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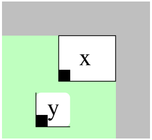

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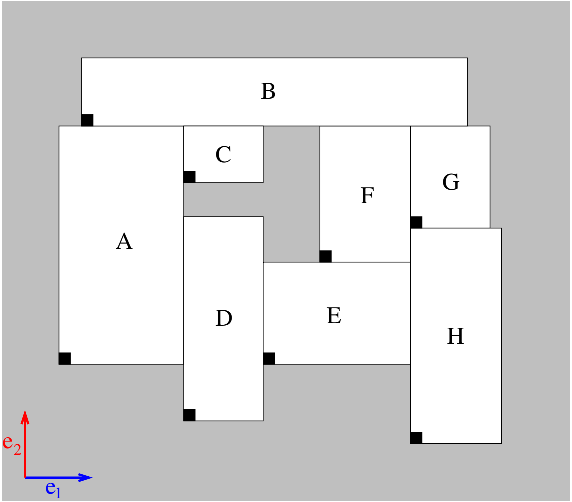

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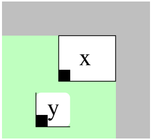

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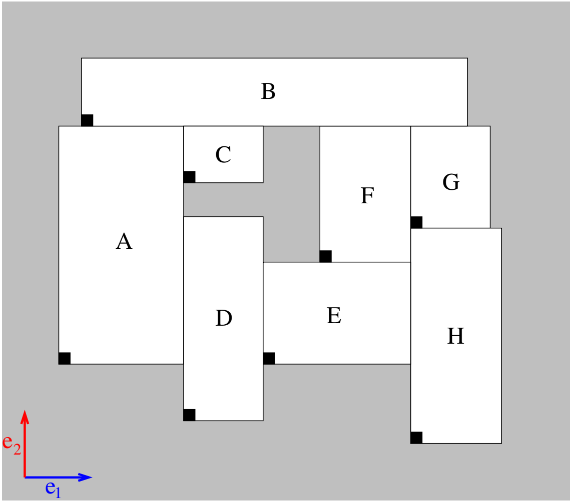

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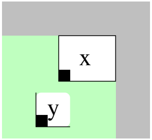

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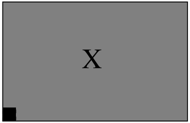

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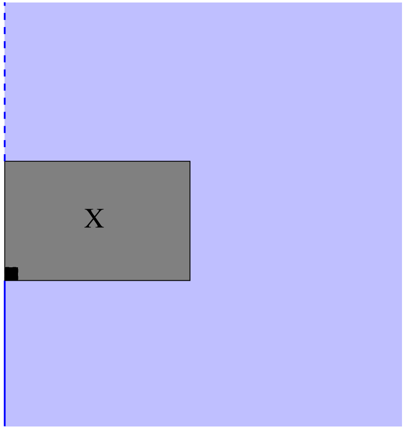

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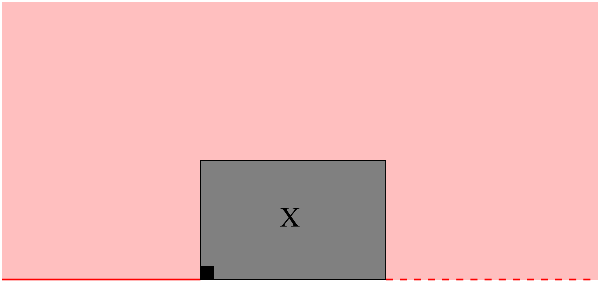

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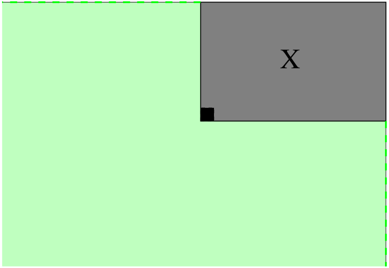

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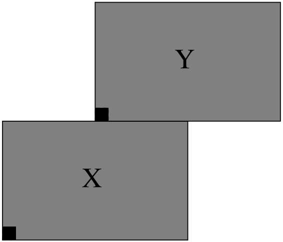

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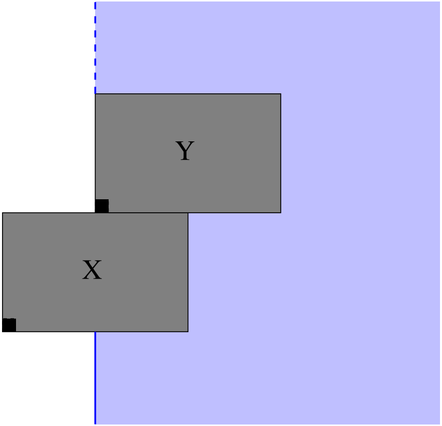

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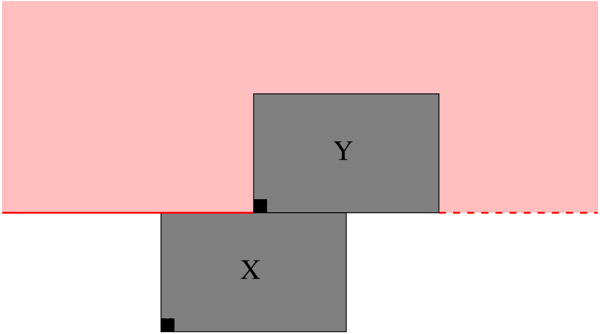

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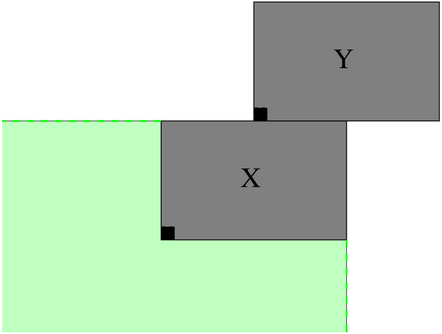

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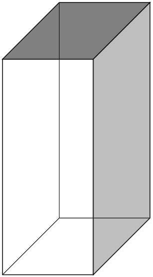

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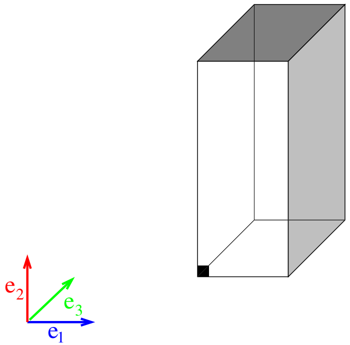

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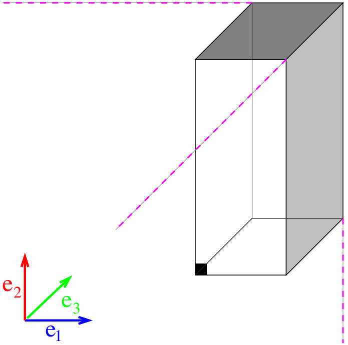

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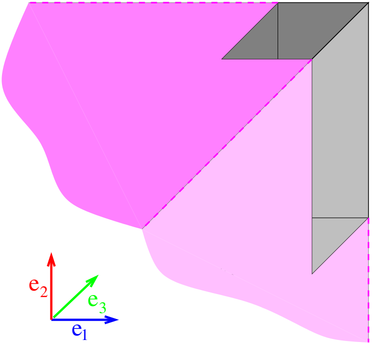

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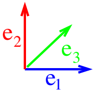

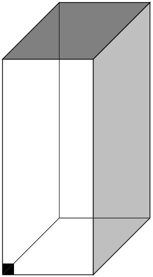

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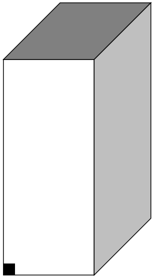

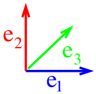

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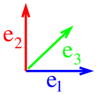

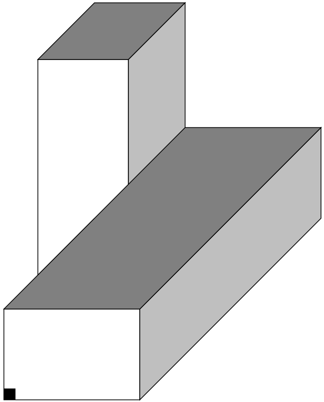

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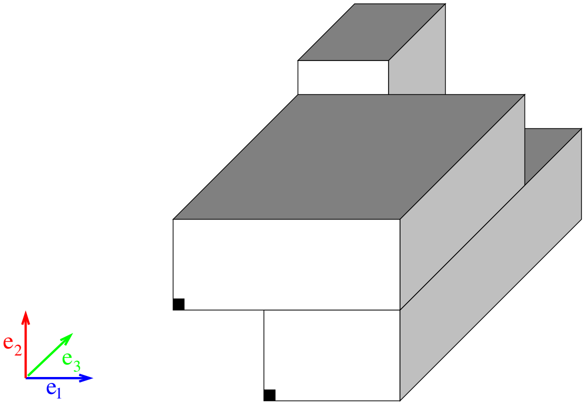

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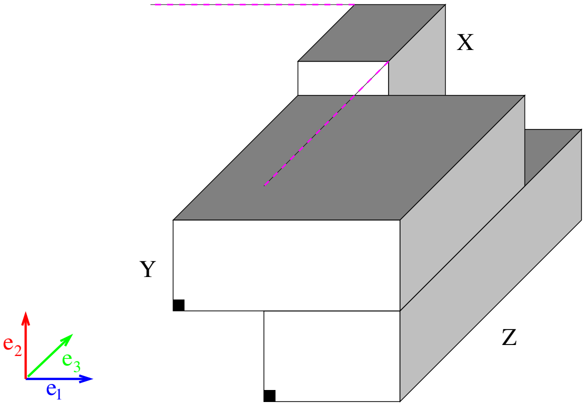

X $\leq_4$ Y

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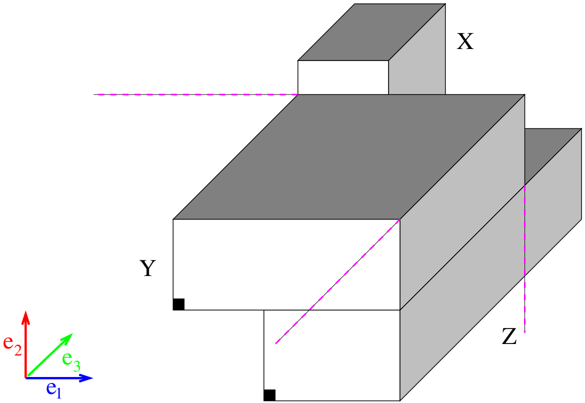

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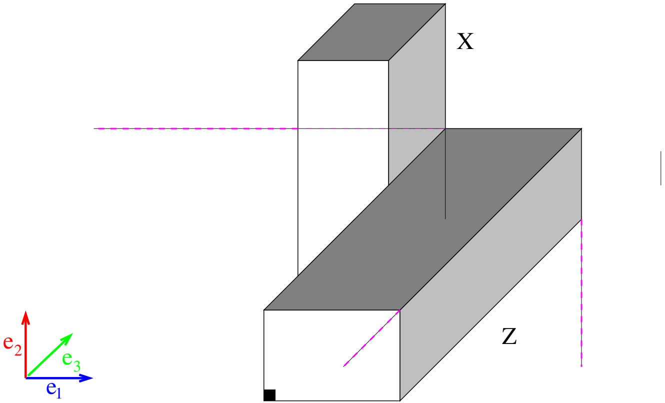

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Main Theorem

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Theorem

For any tiling $\mathscr{T}$ of $\mathbb{R}^d$ with d-dimensional boxes, if $\mathscr{T}$ has only $2d$ infinite boxes and if at most $d+1$ boxes intersect at a point, then $\mathscr{T}$ verifies $dim_{DM}(K(\mathscr{T})) \leq d+1$.

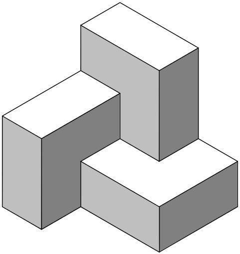
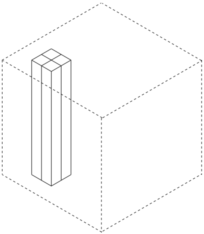
Main Theorem

~~Theorem~~

Any contact system $\mathscr{B}$ of d-dimensional boxes in $\mathbb{R}^d$ with at most $d+1$ boxes intersecting at a point verifies $dim_{DM}(K(\mathscr{B})) \leq d+1$.

Theorem

For any tiling $\mathscr{T}$ of $\mathbb{R}^d$ with d-dimensional boxes, if $\mathscr{T}$ has only $2d$ infinite boxes and if at most $d+1$ boxes intersect at a point, then $\mathscr{T}$ verifies $dim_{DM}(K(\mathscr{T})) \leq d+1$.

Thus no , nor  in $\mathscr{T}$.

Proof Ideas

A tiling in $\mathbb{R}^d$ with only $2d$ infinite boxes and with at most $d+1$ boxes at each point is a **proper d-tiling**.

Theorem

A d-tiling with $2d$ infinite boxes is proper
$\iff$ any 2 intersecting boxes intresect on a $(d-1)$-box.

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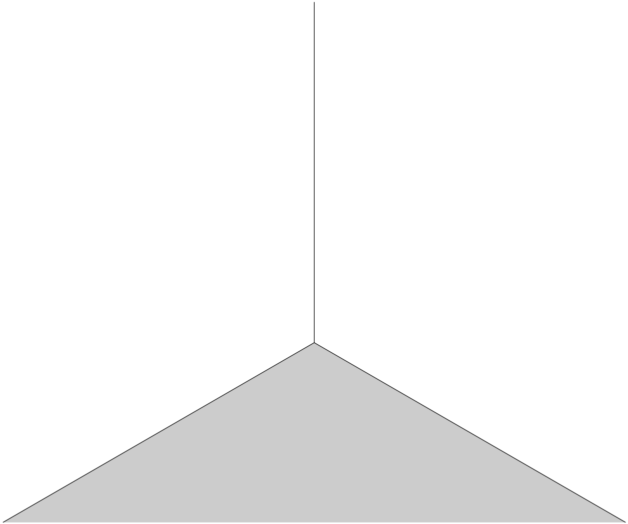
Two sides s and s' are **connected** if there is a sequence $s = s_1, \ldots, s_t = s'$ s.t. $s_i \cap s_{i+1}$ is a $(d-1)$-box.

Lemma

Every maximal set of connected sides induces a $(d-1)$-box.

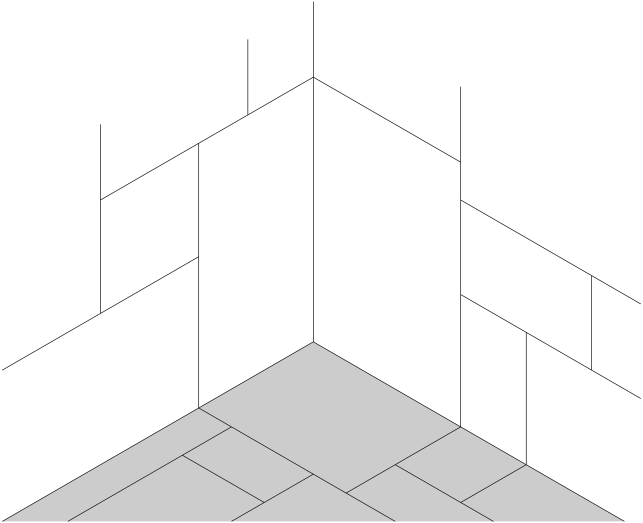
Proof Ideas : Acyclicity of $\leq_{d+1}$

1. Forget the $2d$ infinite boxes.



Proof Ideas : Acyclicity of $\leq_{d+1}$

1. Forget the $2d$ infinite boxes.
2. Consider the highest box X in $\leq_{d+1}$ & Remove it.



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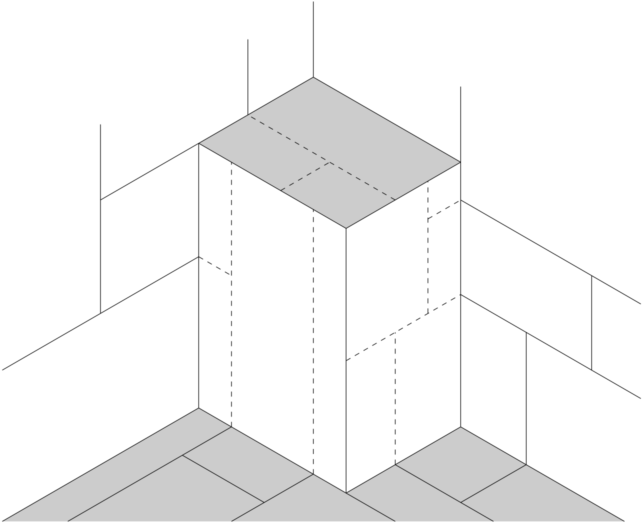

Proof Ideas : Acyclicity of $\leq_{d+1}$

1. Forget the $2d$ infinite boxes.
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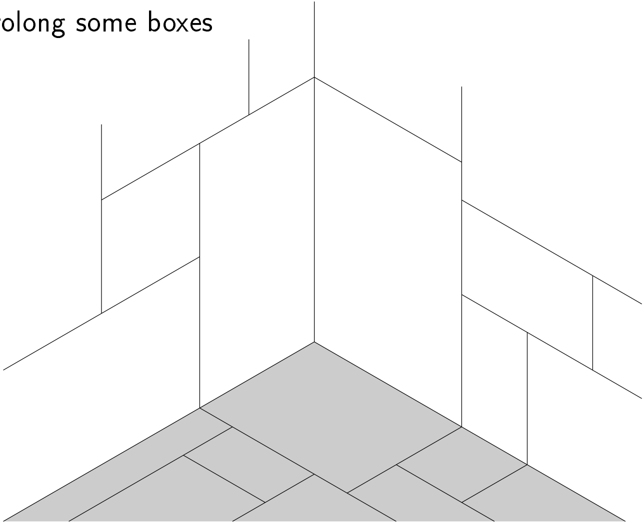
3. Prolong some boxes



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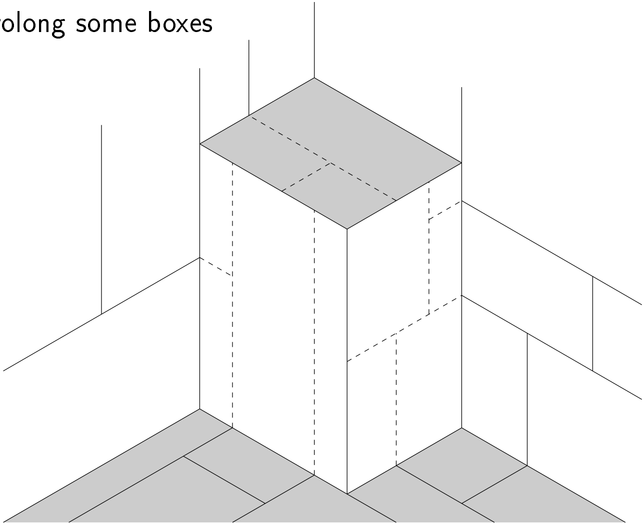

Open Problems

Question 1

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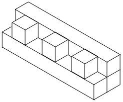
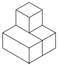
- There exists contact systems $\mathscr{C}$ such that at most $d+1$ boxes intersect at a point but such that $dim_{DM}(K(\mathscr{C})) > d+1$.
- There exists contact systems $\mathscr{C}$ such that the intersection of any k boxes is either empty or it has dimension $d+1-k$, and such that $\mathscr{C}$ is not contained in any proper d-tiling.





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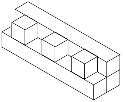

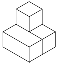

Question 2

Given a contact systems $\mathscr{C}$ such that the intersection of any k boxes is either empty or it has dimension $d+1-k$, does $dim_{DM}(K(\mathscr{C})) \leq d+1$.

More Open Problems

Theorem (Thomassen '84)

K is a clique complex such that $dim_{DM}(K) \leq 3 \iff$ there exists a contact system $\mathscr{C}$ in $\mathbb{R}^2$ with no 4 boxes intersecting and such that $K = K(\mathscr{C})$.

Question 3 : Does this holds ?

K is a clique complex such that $dim_{DM}(K) \leq d+1 \iff$ there exists a contact system $\mathscr{C}$ in $\mathbb{R}^d$ with no $(d+2)$ boxes intersecting and such that $K = K(\mathscr{C})$.

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Question 4 : How to generalize the following Theorem?

Theorem (Hartman et al. '91, and de Fraysseix et al. '94)

G is planar and bipartite $\iff$ G admits a contact system with horizontal or vertical lines.