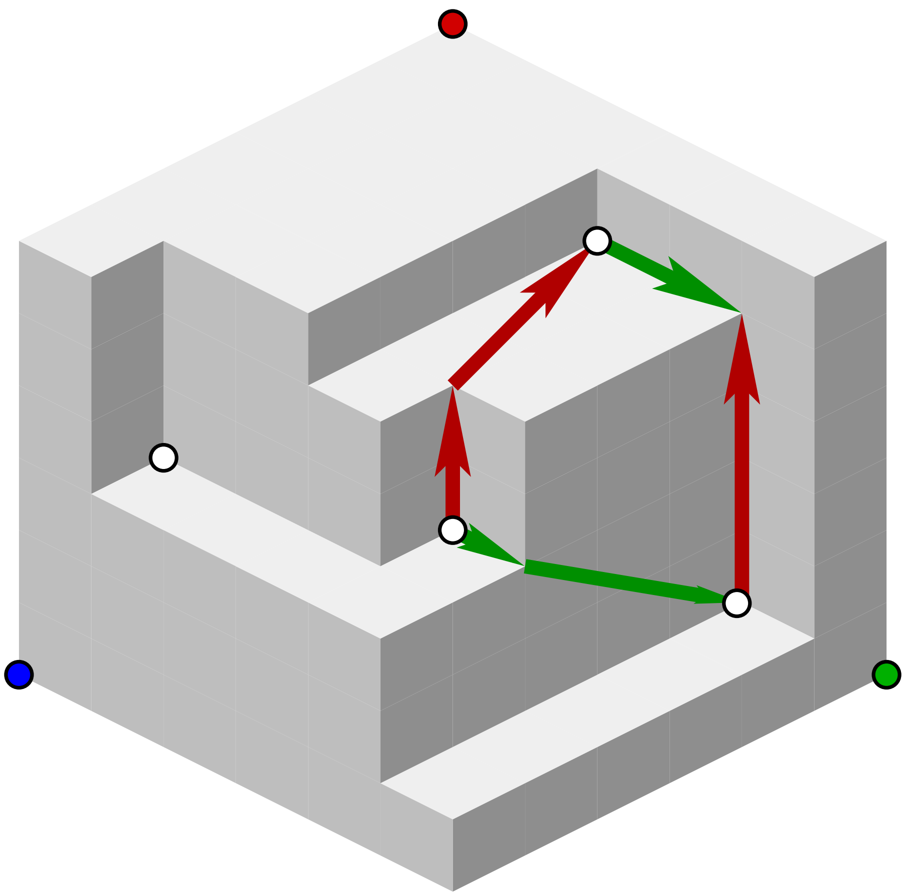
Order & Geometry Workshop

Problem Booklet



Ciążeń Palace, September 19–22, 2018

Preface

The workshop was a gathering of 24 people working in combinatorics and theoretical computer science with special emphasis on discrete geometry, partially ordered sets and all kinds of interplay between them. The workshop was problem oriented. This booklet contains problems presented by participants together with progress notes.

Organizers

- Stefan Felsner, Technische Universität Berlin
- Piotr Micek, Jagiellonian University, Kraków

Webpage

- `http://orderandgeometry2018.tcs.uj.edu.pl`

Former events

- Order & Geometry Workshop, Gułtowy Palace, August 14–17, 2016 (`http://orderandgeometry2016.tcs.uj.edu.pl`)
- Order & Geometry Workshop, Berlin, August 12–15, 2013

Participants

Oswin Aichholzer, Jean Cardinal, Vida Dujmović, Stefan Felsner, Daniel Gonçalves, Grzegorz Gutowski, Penny Haxell, Tony Huynh, Gwenaël Joret, Kolja Knauer, Tamás Mészáros, Piotr Micek, Pat Morin, Torsten Mütze, Patrice Ossona de Mendez, Günter Rote, Felix Schröder, Michał Seweryn, Raphael Steiner, Tom Trotter, Torsten Ueckerdt, Pavel Valtr, Birgit Vogtenhuber, Bartosz Walczak

Scribe

- Michał Seweryn

Problem 1

proposed by Penny Haxell

The *list dimension* of a graph G is the least integer d such that for every integer m and every assignment of lists $A(e) \subseteq \{1, 2, \ldots, m\}$ to the edges of G, if all lists have size at least d, then there exist linear orders $L_1, L_2, \ldots, L_m$ of $V(G)$ such that for every edge $xy \in E(G)$ and every vertex $z \in V(G) \setminus \{x, y\}$ there exists $i \in A(xy)$ such that z is above x and y in L_i.

Question 1.1. *Is list dimension of planar graphs bounded?*

Progress. Penny and Stefan showed that list dimension of paths is at least 3. Bartosz showed that it is at most 3.

Bartosz also showed that list dimension is unbounded already for forests; in fact every graph with two vertices of degree at least d has list dimension $\Theta(\log d)$. On the other hand, if all but at most one vertex of a graph have degree at most d, then the list dimension is $O(d^{\log \log d})$. This shows that list dimension is tied with the second largest degree of a vertex in a graph.

Penny and Patrice generalized the result to containment orders:

Let $(P, \leq)$ be a poset. Let U be the set of all minimal elements of P, and for $x \in P$ let $M(x) = \{v \in U \mid v \leq x\}$. The poset P is a *containment order* if for any two points x, y we have $x \leq_P y$ if and only if $M(x) \subseteq M(y)$. Let $B \subseteq P$, and let $\lambda : B \to \mathcal{P}([m])$ be a list assignment to all elements of B. For $v \in U$ we define the *degree* of v in B by

$$\mathrm{D}(v) = \big|\{\lambda(x) \mid x \in B \text{ and } x \geq v\}\big|.$$

The result is the following:

Let $(P, \leq)$ be a containment order, let $B \subseteq P$, and let λ be a list assignment to B. Let $\Delta_1 = \max_{v \in U} \mathrm{D}(v)$ and let $\Delta_2 = \min_{z \in U} \max_{v \in U \setminus z} \mathrm{D}(v)$. Suppose that

$$\min_{x \in B} |\lambda(x)| \geq \min\Big(\frac{\Delta_1^{\dim P} - 1}{\Delta_1 - 1}, \frac{\Delta_2^{\dim P + 1} - 1}{\Delta_2 - 1}\Big).$$

Then there exist m linear extensions $L_1, \ldots, L_m$ of P such that for every incomparable pair u, v and every $x \in B$, if $x \geq u$, then for some $i \in \lambda(x)$ we have $u <_{L_i} v$.

Problem 2

proposed by Stefan Felsner

Theorem (Schnyder)**.** *The dimension of the incidence order of a planar graph is at most* 3.

Theorem (Brightwell and Trotter)**.** *The dimension of the containment order of vertices and bounded faces of a 3-connected planar graph is at most* 3*.*

Theorem (Brightwell and Trotter)**.** *The dimension of the containment order of vertices and all faces of a 3-connected planar graph is exactly* 4*.*

Theorem (Felsner, Mustaţă, and Pergel)**.** *Deciding if* $\dim(P) \leq 3$ *for the class of orders P of height* 2*, max degree* 5*, and large girth is NP-complete.*

Question 2.1. *What is the complexity of deciding if* $\dim(P) \leq 3$ *for the class of orders P of height* 2 *and max degree* 3*?*

Question 2.2. *What is the complexity of deciding if* $\dim(P) \leq 3$ *for the class of orders P of height* 2 *whose cover graph is planar?*

Remark. An order P is of dimension at most 3 if and only if P can be represented as the containment order of a set of homothetic triangles in the plane, see Figure 1.

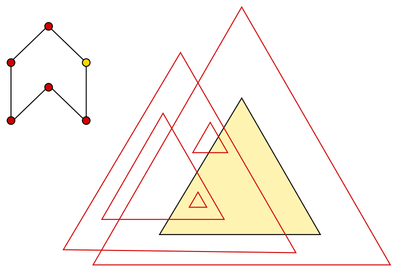


FIGURE 1. A poset with a triangle containment representation.

Problem 3

proposed by Tom Trotter

A class of posets is dim-*bounded*, if dimension of every poset in the class is bounded by a function of the size of a largest standard example in the poset.

Question 3.1. *Is the class of planar graphs* dim-*bounded?*

The second question also regards planar posets, but is only loosely related to the first one.

Question 3.2. *Is Boolean dimension of a planar posets bounded by a constant?*

Progress. Gwenaël, Tamás, Piotr, Tom and Bartosz showed that for every $k \geq 1$, the class of posets whose cover graphs have pathwidth at most k is dim-bounded.

Problem 4

proposed by Stéphan Thomassé & Patrice Ossona de Mendez

Question 4.1. *Is there a function f such that the following holds for every height-2 poset P: If the cover graph G of P has no C_4 subgraph then it has a vertex of degree at most $f(\dim(P))$?*

Since $\dim(P) \leq \text{box}(G) \leq 2\dim(P)$ for height-2 posets, where $\text{box}(G)$ denotes the boxicity of G, this question is equivalent to whether bipartite graphs with no C_4 subgraph have their minimum degree upper bounded by a function of the boxicity.

Progress. Patrice showed that the answer is "yes" if $\dim(P) \leq 3$, and Torsten Ueckerdt showed that the answer is yes if $\text{box}(G) \leq 2$.

In the general case the answer is "no". Torsten constructed for every $d \geq 1$ a poset of height 2 and dimension at most 4, whose cover graph is a bipartite d-regular graph of boxicity 3. This shows that the above results are tight.

Problem 5

proposed by Pat Morin

This problem deals with showing the existence of a special type of Fáry embedding of planar graphs.

For distinct points $p, q \in \mathbb{R}^2$, we say that p *(i,k)-dominates* q if we draw a regular $2k$-gon centered at p and the ray originating at p and containing q intersects the ith edge of this $2k$-gon. We write this as $q \prec_{i,k} p$. (There is a small ambiguity here when the ray passes through a vertex of the $2k$-gon, treat each edge of the $2k$-gon as contain its counterclockwise endpoint but not its clockwise endpoint.)

We say that a non-crossing plane straight-line drawing of a graph G is k-transitive if, for every $i \in \{1, \ldots, k\}$ and every path xyz in G for which $x \prec_{i,k} y \prec_{i,k} z$, the edge xz is also in G. Intuitively, this captures the idea that if xz is not in G then the path xyz should "bend significantly" at y.

Question 5.1. *Does there exist a constant k such that every planar graph G has a k-transitive non-crossing plane straight line drawing?*

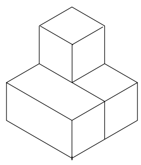
Problem 6

proposed by Daniel Gonçalves

A *box* is the Cartesian product of d closed intervals. A *proper d-tiling $\mathcal{T}$* is a finite collection of interior disjoint d-dimensional boxes that tile $\mathbb{R}^d$, with exactly $2d$ infinite boxes, and such that the intersection of two boxes is either empty or a $(d-1)$-dimensional box. Actually in a proper d-tiling k interesecting boxes intersect on a $(d+1-k)$-dimensional box. The case $d \leq 2$ led us to the following question:

Question 6.1. *Consider a finite set $\mathcal{C}$ of finite d-dimensional boxes such that for every set $\mathcal{B} \subset \mathcal{C}$ of pairwise intersecting boxes, we have that* $\dim(\cap_{B \in \mathcal{B}} B) = d + 1 - |\mathcal{B}|$. *Is $\mathcal{C}$ always a subset of a proper d-tiling ?*

Progress. Trivial examples show that this does not hold.



Felix and Günter found a reduction, which seems to show that the problem of determining, whether a system of boxes can be extended to a proper tiling is NP-hard. There are some technical details that remain to be proved.

Problem 7

proposed by Grzegorz Gutowski

Let G_1 and G_2 be two planar graphs on the same set of vertices.

Question 7.1. *How hard is is to determine, whether there exist drawings of the graphs G_1 and G_2, that agree on the intersection $G_1 \cap G_2$?*

Remark. A similar problem for three graphs is NP-hard.

Problem 8

proposed by Bartosz Walczak & Gwenaël Joret

Theorem (Kozik, Micek, Trotter; unpublished)**.** *The dimension of any planar poset of height h is $O(h^5)$.*

Theorem (Pilipczuk, Siebertz)**.** *The vertex set of every planar graph can be partitioned into a set $\mathcal{P}$ of geodesic paths such that the graph $G/\mathcal{P}$ obtained by contracting the paths in $\mathcal{P}$ has treewidth at most* 8.

Question 8.1. *Can you use the above result to obtain an alternative proof that posets with planar cover graphs have dimension bounded by a polynomial function of height?*

Problem 9

proposed by Vida Dujmović

Given a plane drawing G of a planar graph, a Jordan curve C is a *proper good* curve if $C(0) = C(1)$ is in the outer face of G and the intersection between C and each edge e of G is either empty, a single point, or the entire edge e. Length of a proper good curve is the number of vertices of G it intersects.

Theorem (Bose et al.). *Every n-vertex plane triangulation has a proper good curve of length $\Omega(\sqrt{n})$.*

Theorem (Ravsky and Verbitsky). *There are plane triangulations that do not have a proper good curve of length $\Omega(n^{\sigma})$, for $\sigma \geq 0.986$.*

Question 9.1. *Do there exist proper good curves in n-vertex plane triangulations of length greater than $\Omega(\sqrt{n})$?*

Progress. Vida and Pat showed that every triangulation of maximum degree Δ has a proper good curve of length $\Omega(n^{0.8}/\Delta^2)$.

Problem 10

proposed by Jean Cardinal

Given a vector $\alpha \in \mathbb{N}^n$ and a graph G on n vertices, an *α-orientation* of G is an orientation in which the outdegree of vertex i is exactly α_i. The α-orientations of a graph are connected by operations consisting of reversing the orientation of all edges in a directed cycle. One reversal of a directed cycle is called a *flip*.

Question 10.1. *What is the complexity of the following problem: given a graph G, a vector α, two α-orientations of G, and an integer k, can we transform one orientation into the other in at most k flips?*

Progress. Oswin, Jean, Grzegorz, Birgit, Kolja and Raphael proved that the problem is NP-complete, even in the special case where the α-orientations correspond to perfect matchings in planar, bipartite graphs of maximum degree three and $k = 2$.

Problem 11

proposed by Kolja Knauer

Let $P = (X, \leq)$ and $Q = (X, \leq')$ be two posets on the same set. What is the largest induced acyclic subrelation (or subposet) of the union $P \cup Q = (X, \leq \cup \leq')$? In other words:

Question 11.1. *If we take the union of two acyclic transitive digraphs, what is the largest induced acyclic (transitive) subgraph?*

It is easy to solve this problem, if one of the orders is total. Probably even if one of the orders has bounded width, then it can be done in polynomial time. The general problem of finding the largest induced acyclic subgraph of a digraph is NP-complete and every digraph can easily be written as the (disjoint) union of two acyclic digraphs. So in the vicinity of trying to prove hardness in the above question, two possible nice things to look at would be.

Question 11.2. *What is the smallest number of acyclic transitive digraphs needed to cover the edges of a given digraph?*

Question 11.3. *How hard is it to find the largest induced acyclic transitive subdigraph of a given digraph?*

The first question was told to me by Basile Coëtoux.

Problem 12

proposed by Oswin Aichholzer

A *good drawing* is a drawing of a graph such that any two edges intersect at most once, either at a common endpoint or at a proper crossing, and no three edges cross at a common point.

Question 12.1. *Does every good drawing of K_n contain a non-self-intersecting Hamiltonian cycle?*

Remark. The answer is "yes" for $n \leq 9$.

Problem 13

proposed by Birgit Vogtenhuber

Let S be a set of n points in the plane in general position, i.e., no three points are on a line. Consider the drawing of the complete geometric graph $K_n(S)$ which has the points of S as vertices. We want to color the edges red and blue in a way that the number of monochromatic crossings (that is, crossings between edges of the same color) is minimized.

Question 13.1. *How fast can we find a best two-coloring of the edges for a given set S? What if we allow $c > 2$ colors?*

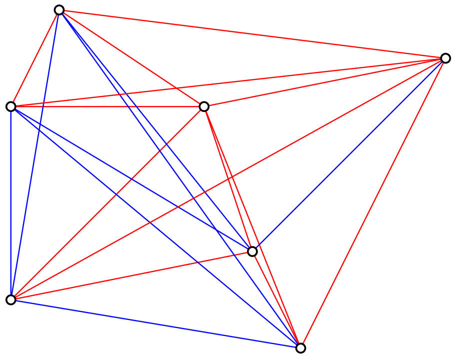


FIGURE 2. A drawing of K_7 with non-optimal edge-coloring and 6 monochromatic crossings.

Remark. The question is identical to finding a maximum cut in the intersection graph of the edges of $K_n(S)$. For general geometric graphs, the problem is known to be NP-complete, even if the underlying point set S is in convex position.

Problem 14

proposed by Kolja Knauer

By considering the dual of a counterexample to Tait's conjecture, we can easily verify the following.

Theorem. *There exists a planar graph G for which there is no partition $V(G) = A \cup B$ such that each of the sets A and B induces a forest.*

The following result can be extracted from a paper of Mohar and Li about dichromatic number of planar digraphs of digirth 4.

Theorem. *For every planar graph G there is a partition $V(G) = A \cup B$ such that each of the sets A and B induces a chordal graph.*

Question 14.1. *Does every planar graph admit a partition $V(G) = A \cup B$ such that A induces a forest and B induces a chordal graph?*

Problem 15

proposed by Piotr Micek

The *queue number* of a poset P (denoted $\text{qn}(P)$) is the least integer d for which there exists a linear extension L of P and a coloring of the edges of the cover graph of P with d colors such that there are no two edges ad and bc of the same color with $a < b < c < d$ in L.

Question 15.1. *Is it true that for any planar poset P the queue number of P is at most the width of P?*

The best known bound so far is $\mathrm{qn}(P) \leq 3w - 2$ by Knauer, Micek and Ueckerdt where P is a planar poset of width w.

Bonus problem

proposed by Bartosz Walczak

Quoridor is a board game for two players. For the detailed decription of the rules see `en.wikipedia.org/wiki/Quoridor`.

Question. *In the game of Quoridor, can there be a position such that no player has a winning strategy?*

Progress. Surprisingly, the answer is "yes". Pavel found a position on the board, for which there is no winning strategy for none of the players.